Measure, Integration, Complex Analysis

Jonathan Ma
johnma@udel.edu

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Contents

Preface

1 Chapter 1. Measure

1.1 Lecture 1. Riemann Integral Review

Intuitively, given some bounded function $f : [a, b] \to \mathbb{R}$, the integral

$$\int_a^b f(x)\, dx$$

expresses the area under the graph of the function. We learned of the Riemann integral in 600. Let's talk about why the theory of Riemann integration can be insufficient.

Example 1.1. Continuous functions are Riemann integrable, but consider the Dirichlet function $\chi_{\mathbb{Q}} : [0, 1] \to \mathbb{R}$, with

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

The lower sums and upper sums of over any partition over $[0, 1]$ of χ_Q never agree, due to the density of the rationals in $\mathbb{R}$, so $\chi_{\mathbb{Q}}$ is not Riemann integrable. On the other hand, consider the Riemann function $\rho : [0, 1] \to \mathbb{R}$, with

$$\rho(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, (p, q) = 1 \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

We can show ρ to be Riemann integrable, with integral 0.

So how can we determine exactly when a function is Riemann integrable? We notice that the Direchlet function is continuous nowhere on $[0, 1]$. However, with the Riemann function, we see that the function is only discontinuous on the rationals points in $[0, 1]$, while being continuous in the irrational points. Later we will formally connect this idea of continuity on certain sets with determining the Riemann integrability of a function. We recall two definitions from 600.

Definition 1.2 (Pointwise convergence)**.** Consider $\{f_n\}$, where $\forall n \in \mathbb{Z}^+$, $f_n : [a, b] \to \mathbb{R}$, and $f : [a, b] \to \mathbb{R}$. We say $f_n \to f$ pointwise if

$$\forall x \in [a, b], \lim_{n\to\infty} f_n(x) = f(x).$$

Definition 1.3 (Uniform convergence)**.** We say $f_n \to f$ uniformly if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{Z}^+$ such that

$$|f(x) - f_n(x)| < \varepsilon, \forall x \in [a, b], \forall n \geq n_0.$$

Equivalently, for a function $g : [a, b] \to \mathbb{R}$, let $\|g\|_\infty := \sup_{a\leq x\leq b} |g(x)|$. Then $f_n \to f$ uniformly iff $\|f_n - f\|_\infty \to 0$.

Theorem 1.4. *If $\{f_n\}$ is a sequence of Riemann integrable functions, and $f_n \to f$ uniformly, then f is Riemann integrable, and*

$$\lim_{n\to\infty} \int f_n = \int f.$$

The same is not true for pointwise convergence. When this convergence is pointwise, and it is true that f is Riemann integrable, it may not be true that their integrals are the same.

Example 1.5. Let $\chi_n : [a,b] \to \mathbb{R}$ be such that

$$\chi_n(x) = \begin{cases} 1 & x = \frac{p}{q}, q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then we can see χ_n is Riemann integrable, (since its set of discontinuities is finite), and $\chi_n \to \chi_{\mathbb{Q}}$. But $\chi_{\mathbb{Q}}$ is not Riemann integrable.

Example 1.6. Consider $f : [a,b] \to \mathbb{R}$ to be the everywhere 0 function. Let $f_n : [a,b] \to \mathbb{R}$ such that

$$f_n(x) = \begin{cases} n & x \in (0, \frac{1}{n}] \\ 0 & x \notin [0, \frac{1}{n}]. \end{cases}$$

Each f_n is Riemann integrable, and $f_n \to f$ pointwise. But the integral of f_n over $[a,b]$ is 1, but the integral of f is 0.

Mainly we'll spend much of this chapter developing the theory of an integral which plays well with pointwise convergence.

1.2 Lecture 2. Weierstrauss Approximation Theorem

Briefly, let us talk about some tools we will need for our discourse. We will spend some time building the Weierstrauss approximation theorem.

Definition 1.7. Let $C[a,b] = \{f : [a,b] \to \mathbb{C} \mid f \text{ is continuous}\}$. Let $C_{\mathbb{R}}[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$. Note, for any $f : [a,b] \to \mathbb{C}$, we can split f into its real and imaginary parts, $f = \operatorname{Re} f + i \operatorname{Im} f$. If f is continuous, then its real and imaginary parts are also continuous.

Definition 1.8. Given $f \in C[a,b]$, or $f \in C_{\mathbb{R}}[a,b]$, we define

$$\|f\|_\infty := \sup_{a \leq x \leq b} |f(x)|,$$

to be the uniform, or infinity norm of f.

Recall that $C[a,b]$ is a vector space over $\mathbb{C}$.

Theorem 1.9. $\|\cdot\|_\infty$ *is a norm.*

Proof. If $\|f\|_\infty$ is zero, then by definition, f must be identically zero. Also properties of absolute value / modulus, $|\lambda f(x)| = |\lambda||f(x)|$. The triangle inequality also follows as a result of the triangle inequality of the absolute value / modulus. □

Definition 1.10. A pair $(X, \|\cdot\|)$, where X is a vector space, and $\|\cdot\| : X \to \mathbb{R}$ is a norm, is called a normed vector space.

Proposition 1.11. *Given a normed vector space* $(X, \|\cdot\|)$*, we can define a metric* d *on* X *by letting* $d(f,g) = \|f - g\|$.

All of the properties of metrics follow by the properties of norms.

Definition 1.12. Recall if (X, d) is a metric space, and suppose $\{x_n\} \subset X$. We say $x_n \to x$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}^+$ such that

$$d(x_n, x) < \varepsilon, \quad \forall n \geq N.$$

The main point of this review is to answer the following questions.

1. Can continuous functions be approximated well in uniform distance by some class of familiar functions?

2. How much information for a continuous function $f : [a,b] \to \mathbb{R}$ is encoded in its Riemann integral?

1.2.1 Moments of a Function

Definition 1.13. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. The nth moment of f is the value

$$\mu_n(f) = \int_a^b f(x)x^n \, dx, \quad n = 0, 1, 2, \ldots.$$

This notion of moments is borrowed from probability theory. To help answer our second question from above, we ask the following. If $f : [a,b] \to \mathbb{R}$ is a continuous function, does the sequence $\{\mu_n(f)\}$ determine f?

Proposition 1.14. *In other words, if* $\mu_n(f) = \mu_n(g), \forall n \in \mathbb{Z}^+$*, is it true that* $f = g$*?*

The answer is yes. The moments of a function f determine the function f. To prove this, we will use the following result.

Theorem 1.15 (Weierstrauss approximation theorem)**.** *For every* $f \in C[a,b]$*, there exists a sequence of polynomials* $\{P_n\}$ *such that* $\|P_n - f\|_\infty \to 0$ *uniformly.*

This theorem effectively answers our first question. Before we prove this, let us see why this answers our second question.

Proof of Proposition 1.14. Let $f, g \in C[a,b]$. We claim that if we show that when $\mu_n(f) = 0, \forall n \in \mathbb{Z}^+$, then $f = 0$ identically, this is enough to prove that $\mu_n(f) = \mu_n(g), \forall n \in \mathbb{Z}^+ \implies f = g$. This is because if $\mu_n(f) = \mu_n(g), \forall n \in \mathbb{Z}^+$, then

$$\mu_n(f) - \mu_n(g) = 0 \implies \mu_n(f - g) = 0 \implies f - g = 0 \implies f = g.$$

Suppose $\mu_n(f) = 0, \forall n \in \mathbb{Z}^+$. Then

$$\int_a^b f(x)x^n \, dx = 0.$$

But this implies that for any polynomial $P(x)$,

$$\int_a^b f(x)P(x) \, dx = 0.$$

By the Weierstrauss approximation theorem, there is some sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly. Then

$$\int_a^b f(x)P_n(x)\,dx = 0$$
$$\implies \lim_{n\to\infty} \int_a^b f(x)P_n(x)\,dx = \int_a^b (f(x))^2\,dx = 0.$$

Since $(f(x))^2$ is nonnegative, and its integral is 0, we conclude that f is identically 0. $\square$

With respect to answering our second question, this means that we can recover a function from its moments. Speaking in an applied sense, we see this as being useful in signal processing, where we can reconstruct a signal using a finite set of moments.

1.2.2 Weierstrauss Approximation Theorem

The proofs for the Weierstrauss approximation, and the following Stone-Weierstrauss approximation theorem are in Rudin's *Principles of Mathematical Analysis.*

1.3 Lecture 3. Stone-Weierstrauss Approximation Theorem

Assume that K is a compact subset of a metric space (X, d). Consider $C(K)$, the space of all continuous complex-valued functions on K, and write $C_{\mathbb{R}}(K)$ to be the space of continuous real-valued functions on K.

1.3.1 Stone-Weierstrauss Approximation Theorem

Theorem 1.16 (Stone-Weierstrauss, real case)**.** *Let K be as above, and let $A \subset C_{\mathbb{R}}(K)$ be an algebra:*

1. $f, g \in A \implies fg \in A$,
2. $f, g \in A \implies f + g \in A$,
3. $\lambda f \in A, \forall \lambda \in \mathbb{R}, f \in A$.

Suppose A separates points: given $x \neq y$, $\exists f \in A, f(x) \neq f(y)$, and that A does not vanish: $\forall x \in K, \exists f \in A, f(x) \neq 0$. Then A is dense in $C_{\mathbb{R}}(K)$ in $\|\cdot\|_\infty$.

It is not hard to see that the Weierstrauss approximation theorem follows from the Stone-Weierstrauss approximation theorem. Let us see some examples of the Stone-Weierstrauss approximation theorem applied.

Example 1.17. Let K, L be compact sets, and for $f \in C(K), g \in C(L)$, let $f \otimes g : K \times L \to \mathbb{R}$ be given by

$$(f \otimes g)(x, y) = f(x)g(y).$$

Then $f \otimes g \in C_{\mathbb{R}}(K \times L)$. Consider $A = \text{span}\{f \otimes g \mid f \in C_{\mathbb{R}}(K), g \in C_{\mathbb{R}}(L)\}$. Note that A satisfies all the assumptions of the Stone-Weierstrauss theorem. Consider that

$$(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 g_1 f_2 g_2 = (f_1 f_2) \otimes (g_1 g_2).$$

We can see A separates points, where we can take the tensor product of the identity with itself to get the identity, and because $1 \otimes 1 = 1$, we see A does not vanish. Therefore, A is dense in $C_{\mathbb{R}}(K \times L)$.

We will now state the complex-valued version of the Stone-Weierstrauss approximation theorem.

Theorem 1.18 (Stone-Weierstrauss approximation theorem, complex case)**.** *We take the same assumptions as the real case, but we add the assumption that A is a self-adjoint algebra, that is $\forall f \in A, \overline{f} \in A$. Then A is dense in $C(K)$.*

Example 1.19. This result gets used in Fourier analysis. Suppose

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Then $\mathbb{T}$ is a compact metric space under the arclength metric. Consider $C(\mathbb{T})$. The elementary frequencies, or trigonometric polynomials are defined as follows. For $n \in \mathbb{Z}$, let $\zeta_n \in C(\mathbb{T})$,

$$\zeta_n(z) = z^n.$$

Why are these ζ_n called trigonometric functions? Write $z = e^{i\theta}, \theta \in [0, 2\pi)$. Then $e^{i\theta} = \cos\theta + i\sin\theta$. Then we see

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta.$$

Let $A = \operatorname{span}\{\zeta_n \mid n \in \mathbb{Z}\}$. Then we claim A satisfies the conditions of the Stone-Weierstrauss approximation theorem in the complex case. The significance of this is that the elementary functions are enough to recover all continuous functions in $C(\mathbb{T})$, a key fact in harmonic and Fourier analysis.

1.3.2 Introducing the Lebesgue Theory

Suppose that $f : [0, 1] \to \mathbb{R}$. We want to define $\int f$ as a real number equal to the area under curve of f. The Lebesgue theory is developed with the intuition of calculating $\int f$ by dividing the codomain of f into, initially interval partitions, and and summing by the lengths of these partitions times the value of f over that interval. Later on, we generalize this notion of interval partitions into general partitons, and their lengths as measures. To achieve this generalization, we are tasked with developing a notion of size of a subset of $\mathbb{R}$ which is not necessarily an interval.

1.3.3 Constructing a Measure

We would like a function $m : \mathcal{P} \to [0, \pm\infty]$ which gives us the size of a subset of $\mathbb{R}$. It would be really nice if ...

1. m is defined on all of $\mathcal{P}$.

2. If $S = [a, b] \subset \mathbb{R}$, then $m([a, b]) = b - a$.

3. If $S_1, S_2, \dots,$ are all a countable collection of disjoint sets, then we should like to have

$$m\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} m(S_n).$$

4. If $S \subset \mathbb{R}$, we would want m to be translation invariant: for any $a \in \mathbb{R}$, $m(S) = m(\{s + a \in S \mid s \in S\}) =: m(S + a)$

Since this is my second time seeing the construction of this measure, I will spoil this and say this is impossible!

Proposition 1.20. *There exists sets which, given a function m satisfying the 4 properties above, contradict the properties.*

Proof. Assume $m : \mathcal{P}(\mathbb{R}) \to [0, \pm\infty)$ exists. Define an equivalence relation on $\mathbb{R}$ such that $x, y \in \mathbb{R}, x \equiv y$ if $x - y \in \mathbb{Q}$. These equivalence classes are the cosets of $\mathbb{Q}$ by $\mathbb{R}$. Define $N \subset [0,1)$ by selecting one point from every equivalence class. It is a simple argument to see that every equivalence class intersects $[0,1)$. Let $\mathcal{R} = \mathbb{Q} \cap [0,1]$. For $r \in \mathbb{R}$, let

$$N_r = \{x + r \mid x \in N, x < 1 - r\} \cup \{-1 + x + r \mid x \in N, 1 - r \leq x < 1\} \subset [0,1).$$

We claim $[0,1) = \bigcup_{r \in \mathcal{R}} N_r$. Clearly $\bigcup_{r \in \mathcal{R}} \subset [0,1)$ by construction of N_r. Conversely, choose any point $x \in [0,1)$. Now, $(x + \mathbb{Q}) \cap [0,1) = \{y\}, y \in N$. Then $x - y \in \mathbb{Q}$. If $x < y$, $r = y - x \implies x = y + r \in N_r$. If $x > y$, then $-1 + x - y = r \in N_r$. We also claim that each $N_r \cap N_t = \emptyset$, if $r \neq t$. We skip the proof of this claim for brevity. Therefore,

$$1 = m([0,1)) = \sum_{r \in \mathcal{R}} m(N_r) = \sum_{r \in R} m(N),$$

a contradiction. Either this above sum diverges or it is 0.

□

This set N is called a Vitali set. Luckily, we can only *not* satisfy 1, while satisfying $2, 3, 4$. What we will present is what Caratheodory described in his construction of the Lebesgue measure and the Lebesgue σ-algebra, to solve this problem.

1.4 Lecture 4. Caratheodory's Construction

Since we cannot define the desirable measure on the power set of $\mathbb{R}$, let us restrict ourselves. We will now aim to define Lebesgue measureable sets $\mathcal{L}(\mathbb{R})$, and the Lebesgue measure $\lambda : \mathcal{L}(\mathbb{R}) \to [0, \infty]$. We will start with defining the outer measure, and showing that when restricted sufficiently, the outer measure is the measure we desire.

1.4.1 Outer Measure

Definition 1.21 (Outer Measure)**.** For any $A \subset \mathbb{R}$,

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} b_n - a_n \mid A \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

In plain text, the outer measure of a set A is the infimum over the lengths of countable open interval coverings of A.

Example 1.22. The outer measure, for $x \in \mathbb{R}$, of $\{x\}$ is 0. The outer measure of the set $\mathbb{Z}$ is 0. You'd intuitively be able to shrink intervals around each point in these sets smaller and smaller, until they become singleton intervals, which have no mass.

Proposition 1.23. *The outer measure of any countable set is 0.*

Proof. Suppose A is countable, with elements $\{a_n\}$. Then let $\varepsilon > 0$, and consider the open interval covering

$$\left\{ \left(\frac{a_n - \varepsilon}{2^n}, \frac{a_n + \varepsilon}{2^n} \right) \right\}.$$

This is clearly an open interval covering of A, and its length is

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

We send $\varepsilon \to 0$ to conclude our proof. $\square$

Proposition 1.24. *The outer measure of* $\mathbb{R}$ *is* ∞.

Theorem 1.25. *The outer measure satisfies the following.*

1. *If* $A \subset B$, *then* $\lambda^*(A) \leq \lambda^*(B)$.
2. *For any* $A \subset \mathbb{R}$, $\lambda^*(A+t) = \lambda^*(A)$, *for any* $t \in \mathbb{R}$.
3. *For* $\{A_n\}$ *a countable collection of sets of* $\mathbb{R}$, *we have*

$$\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$$

Proof. For 1, if $\{(a_n, b_n)\}$ is an open interval covering of B, then it is an open interval covering of A. For 2, consider that $A \subset \bigcup_{n=1}^{\infty}(a_n, b_n)$ iff $A + t \subset \bigcup_{n=1}^{\infty}(a_n + t, b_n + t)$. Also,

$$\sum_{n=1}^{\infty} b_n - a_n = \sum_{n=1}^{\infty} a_n + t - b_n - t.$$

For 3, let $\varepsilon > 0$. Since $\lambda^*(A_i)$ is an infimum over lengths of open interval covers of A_i, there exists an open interval covering of A_i, such that

$$A_i \subset \bigcup_{k=1}^{\infty} (a_k^{(i)}, b_k^{(i)})$$

such that

$$\lambda^*(A_i) \leq \sum_{k=1}^{\infty} (b_k^{(i)} - a_k^{(i)}) < \lambda^*(A_i) + \frac{\varepsilon}{2^i}.$$

Set $A = \bigcup_{i=1}^{\infty} A_i$. Therefore,

$$A \subset \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} (a_k^{(i)}, b_k^{(i)}).$$

Therefore,

$$\begin{aligned} \lambda^*(A) \leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (b_k^{(i)} - a_k^{(i)}) &\leq \sum_{i=1}^{\infty} \lambda^*(A_i) + \frac{\varepsilon}{2^i} \\ &= \varepsilon + \sum_{i=1}^{\infty} \lambda^*(A_i). \end{aligned}$$

Sending $\varepsilon \to 0$ gives us our result. $\square$

We will now build $\mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ and the Lebesgue measure $\lambda : \mathbb{R} \to [0, \infty]$ to be defined as $\lambda^*|_{\mathcal{L}(\mathbb{R})}$.

1.4.2 Lebesgue Measurable Sets

Definition 1.26. A set $E \subset \mathbb{R}$ is called Lebesgue measurable if $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^C)$, for all $A \subset \mathbb{R}$.

Definition 1.27. Let $\mathcal{L}(\mathbb{R}) = \{E \subset \mathbb{R} \mid E \text{ is Lebesgue measurable}\}$.

Theorem 1.28. *The collection $\mathcal{L}(\mathbb{R})$ of all Lebesgue measurable sets satisfies the following:*

1. $\emptyset \in \mathcal{L}(\mathbb{R})$.
2. *If $E \in \mathcal{L}(\mathbb{R})$, then $E^C \in \mathcal{L}(\mathbb{R})$.*
3. *If $\{E_n\} \subset \mathcal{L}(\mathbb{R})$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{L}(\mathbb{R})$.*
4. *If $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$, then if $\{E_n\} \subset \mathcal{L}(\mathbb{R})$ is a countable and mutually disjoint collection of Lebesgue measurable sets, then*
$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

Proof. For 1, let $A \subset \mathbb{R}$, consider that

$$\lambda^*(A) = \lambda^*(A \cap \emptyset) + \lambda^*(A \cap \mathbb{R}) = \lambda^*(A).$$

For 2, for some $E \in \mathcal{L}(\mathbb{R})$,

$$\lambda^*(A \cap E) + \lambda^*(A \cap E^C) = \lambda^*(A \cap E^C) + \lambda^*(A \cap E) = \lambda^*(A).$$

For 3, assume E_1, E_2 are Lebesgue measurable. Then

$$\begin{aligned} \lambda^*(A) &= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^C) \\ &= \lambda^*(A \cap E_1 \cap E_2) + \lambda^*(A \cap E_1 \cap E_2^C) + \lambda^*(A \cap E_1^C) \\ &\geq \lambda^*(A \cap (E_1 \cap E_2)) + \lambda^*(A \cap (E_1 \cap E_2)^C), \end{aligned}$$

hence $E_1 \cap E_2$ is Lebesgue measurable. But $E_1 \cup E_2 = (E_1^C \cap E_2^C)^C$, so $E_1 \cup E_2$ is Lebesgue measurable. Applying an induction argument will show that $\mathcal{L}(\mathbb{R})$ is closed under finite unions. Now, for 4, consider E_1, E_2 two disjoint Lebesgue measurable sets. Then

$$\lambda(E_1 \cup E_2) = \lambda((E_1 \cup E_2) \cap E_1) + \lambda((E_1 \cup E_2) \cap E_1^C) = \lambda(E_1) + \lambda(E_2).$$

We then show apply an induction argument to show that this holds for all $n \in \mathbb{Z}^+$. Assuming we have done this, let E_n be a countable collection of sets in $\mathcal{L}(E_n)$. We will now show that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{L}(\mathbb{R})$. Let $E = \bigcup_{i=1}^{\infty} E_i$. Set $F_1 = E_1, F_2 = E_2 \setminus F_1, F_3 = E_3 \setminus F_2, \ldots$. Then $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$, and $E = \bigcup_{i=1}^{\infty} F_i$. Fix $A \subset \mathbb{R}$,

$$\begin{aligned} \lambda^*(A) &\geq \lambda^*(A \cap (\cup_{i=1}^{n} F_i)) + \lambda^*(A \cap (\cup_{i=1}^{n} F_i)^C) \\ &\geq \sum_{i=1}^{n} \lambda^*(A \cap F_i) + \lambda^*(A \cap E^C) \\ \implies \lambda^*(A) &\geq \sum_{i=1}^{\infty} \lambda^*(A \cap F_i) + \lambda^*(A \cap E^C) \\ &\geq \lambda^*(A \cap E) + \lambda^*(A \cap E^C). \end{aligned}$$

□

1.5 Lecture 5. Characterization of Riemann Integrable Functions

I've skipped ahead a little, because I've taken a version of this course before, but I'll add some definitions here which are covered in asynchronous lectures.

Definition 1.29. The Borel set $\mathcal{B}$ of $\mathbb{R}$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}$.

Proposition 1.30. *Compact subsets of* $\mathbb{R}$ *have finite measure.*

Now here is a question. If f is continuous, we know f is Riemann integrable. How far away can f be from continuous such that f is still Riemann integrable? Recall that we said a set $E \subset \mathbb{R}$ has measure 0 if $\forall \varepsilon > 0, \exists \{(a_k, b_k)\}_{k=1}^{\infty}$, such that $E \subset \bigcup_{i=1}^{n} \{(a_{k_i}, b_{k_i})\} < \varepsilon$. If $A \subset \mathbb{R}$ is countable, then it has measure 0. The Cantor set, which is uncountable also has measure 0. Given a bounded function $f : [a, b] \to \mathbb{R}$, denote $E(f)$ the set of discontinuities of f. Note that f is continuous iff $E(f) = \emptyset$.

Theorem 1.31. *Let* $f : [a, b] \to \mathbb{R}$ *be bounded. Then* f *is Riemann integrable iff* $\mu(E(f)) = 0$.

Proof. Set $E := E(f)$. Assume that $E(f) = 0$. Recall that f is not continuous at z iff

$$\exists \varepsilon > 0 (\forall \delta > 0 \exists x, y \in [z - \delta, z + \delta](|f(x) - f(y)| \geq \varepsilon)).$$

For $k \in \mathbb{N}$, let

$$E_k = \left\{ z \in [a, b] \mid \forall \delta > 0, \exists x, y \in [z - \delta, z + \delta] \left(|f(x) - f(y)| > \frac{1}{k} \right) \right\}.$$

Then $E = \bigcup_{k=1}^{\infty} E_k$. Also, E_k is closed, which is a claim, but also an exercise. Let $\varepsilon > 0$. Then there exists an open interval covering of E, $\{I_k\}_{k=1}^{\infty}$, such that $\sum_{k=1}^{\infty} \mu(I_k) < \varepsilon$. Fix $k \in \mathbb{N}$ and consider that E_k is closed and bounded, so it is compact, and $E_k \subset \bigcup_{\ell=1}^{\infty} I_k$. So $E_k \subset \bigcup_{\ell=1}^{r} I_\ell$. Consider the set

$$[a, b] \setminus \bigcup_{\ell=1}^{r} I_\ell =: K.$$

Then K is compact, since it is a closed and bounded set. For all $z \in K$, $z \notin E_k$. Therefore, $\exists \delta > 0$ such that $\forall x, y \in [z - \delta, z + \delta]$,

$$|f(x) - f(y)| < \frac{1}{k}.$$

Fix $z \in K$, and now

$$K \subset \bigcup_{\delta > 0} (z - \delta, z + \delta).$$

By compactness,

$$K \subset (z - \Delta, z + \Delta) =: J,$$

where $0 < \Delta < (b - a)$. Consider the intervals

$$I_1, I_2, \ldots, I_r, J.$$

Let P be the partition of $[a, b]$ arising from the endpoints these intervals. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum \left(\sup_{I_n} f - \inf_{I_n} f \right) \lambda(I_n) + \sum \left(\sup_{J} f - \inf_{J} f \right) \lambda(J) \\ &< 2 \|f\|_\infty + \frac{1}{k}(b - a) \\ &< \varepsilon(2 \|f\|_\infty + (b - a)). \end{aligned}$$

Conversely, assume that f is Riemann integrable. Write $E = \bigcup_{j=1}^{\infty} E_k$ as before. It suffices to show that $\lambda(E_k) = 0$, for all $k \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there exists a partition $P = \{I_1, \dots, I_n\}$ such that $U(f, P) - L(f, P) < \varepsilon$. Consider $J = \{I_i \mid I_i \cap E_k \neq \emptyset\}$. Then

$$\sum_{i \in J} \frac{1}{k} \lambda(I_i) \leq U(f, p) - L(f, P) < \frac{\varepsilon}{k},$$

and $E_k \subset \bigcup_{i \in J} I_i$. □

1.6 Lecture 6. Lebesgue Measurable Functions

Recall if $f : A \to B$, then for $U \subset B$, $f^{-1}(U) = \{x \in A \mid f(x) \in U\}$. Also if Y is a collection of sets in B, then

$$f^{-1}\left(\bigcup_{\alpha \in Y} \alpha\right) = \bigcup_{\alpha \in Y} f^{-1}(\alpha),$$

for any $Y \subset B$.

Definition 1.32. Let E be a measurable set. A function $f : E \to \mathbb{R}$ is called measurable if $f^{-1}((\alpha, \infty))$ is measurable, for all $\alpha \in \mathbb{R}$.

Example 1.33. Let E be a measurable set. Let $\chi_E : \mathbb{R} \to \mathbb{R}$ be the characteristic function of E. Then χ_E is measurable. If $\alpha > 1$, $\chi_E^{-1}((\alpha, \infty)) = \emptyset$. If $0 \leq \alpha < 1$, then $\chi_E^{-1}((\alpha, \infty)) = E$. If $\alpha < 0$, then $\chi_E^{-1}((\alpha, \infty)) = \mathbb{R}$.

In fact, for any $E \subset \mathbb{R}$, χ_E is measurable iff E is measurable.

Definition 1.34. A function $f : \mathbb{R} \to \mathbb{R}$ is called simple if there exists measurable disjoint sets $E_i \subset \mathbb{R}$, and $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, such that

$$f = \sum_{i=1}^{n} c_i \chi_{E_i}.$$

Theorem 1.35. *Simple functions are measurable.*

1.7 Lecture 7. Simple Functions

Theorem 1.36. *If $f, g : \mathbb{R} \to \mathbb{R}$ is measurable, then*

1. *$f + g$ is measurable,*
2. *cf is measurable, for all $c \in \mathbb{R}$,*
3. *fg is measurable,*
4. *$\phi \circ f$ is measurable, if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof. We will prove 4. Let $O \subset \mathbb{R}$ be an open set. Then

$$(\phi \circ f)^{-1}(O) = f^{-1}(\phi^{-1}(O)).$$

This is clearly measurable, since ϕ is continuous. □

Theorem 1.37. *Let $\{f_n\}$, $f_n : \mathbb{R} \to [-\infty, \infty]$ be a sequence of measurable functions. Then*

1. $\sup_n f_n$ *is a measurable function,*
2. $\inf_n f_n$ *is a measurable function,*
3. $\liminf_{n\to\infty} f_n, \limsup_{n\to\infty} f_n$ *are measurable.*
4. *If* $\lim_{n\to\infty} f_n(x)$ *exists for all* $x \in \mathbb{R}$*, then* $\lim_{n\to\infty} f_n$ *is measurable.*

Proof. For 1, let $f = \sup_n f_n$. Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f^{-1}((\alpha, \infty)) &= \{x \in \mathbb{R} \mid f(x) > \alpha\} \\ &= \{x \in \mathbb{R} \mid \exists n \in \mathbb{Z}^+, f_n(x) \geq \alpha\} \\ &= \bigcup_{n\in\mathbb{Z}^+} \{x \in \mathbb{R} \mid f_n(x) > \alpha\} \\ &= \bigcup_{n\in\mathbb{Z}^+} f_n^{-1}((\alpha, \infty)). \end{aligned}$$

We also need to check $f^{-1}(\{\infty\})$ is measurable. Consider

$$\begin{aligned} f^{-1}(\{\infty\}) &= \{x \in \mathbb{R} \mid \forall N, \exists n(f_n(x) > N)\} \\ &= \bigcap_{N\in\mathbb{Z}^+} \bigcup_{n\in\mathbb{Z}^+} \{x \in \mathbb{R} \mid (f_n(x) > N)\} \\ &= \bigcap_{N\in\mathbb{Z}^+} \bigcup_{n\in\mathbb{Z}^+} f_n^{-1}((N, \infty)). \end{aligned}$$

For 2, perform the analogous analysis with $(-\infty, \alpha)$. For 3, by part 2, $\inf_{k\geq n} f_k$ is measurable, and by part 1, $\sup_n \inf_{k\geq n} f_k$ is measurable. Lim sup case is done similarly. For 4, if the limit of a sequence exists, it is equal to the lim inf of the sequence. □

Here is the big kicker of the Lebesgue theory.

Theorem 1.38. *Let $f : \mathbb{R} \to [0, \infty]$ be measurable. Then there exists simple functions $f_n : \mathbb{R} \to [-\infty, \infty]$ such that*

$$f_1 \leq f_2 \leq \dots$$

and $\lim_{n\to\infty} f_n = f$*. Furthermore, on bounded sets, this convergence is uniform.*

Proof. For $k = 1, 2, \dots N2^N$, let

$$E_k = \left\{ x \in \mathbb{R} \mid \frac{k-1}{2^N} < f(x) < \frac{k}{2^N} \right\}.$$

Then let $f_N(x) = \sum_{k=1}^{N2^N} \frac{k-1}{2^N} \chi_{E_K}(x)$, whenever $f(x) \leq N$, and $f_N(x) = N$, whenever $f(x) > N$. It is clear that f_N is dominated by f, and that $f_1 \leq f_2 \leq \dots$. It is a good exercise to prove that $f_N \to f$ pointwise. □

As a remark, f is measurable iff it is the limit of simple measurable functions.

1.7.1 Lebesgue Integral of Simple Nonnegative Measurable Functions

Definition 1.39. Let $f : \mathbb{R} \to [0, \infty]$ be a nonnegative simple measurable function, written canonically:

$$f = \sum_{i=1}^{k} c_i \chi_{E_i}.$$

Then define

$$\int f = \sum_{i=1}^{k} c_i \mu(E_i).$$

If A is measurable, let

$$\int_A f = \int f \chi_A.$$

Definition 1.40. Let $f : A \to [0, \infty]$ be a measurable function. Define

$$\int_A f = \sup \left\{ \int_A g \mid g \text{ is simple, and } 0 \le g \le f \right\}.$$

Theorem 1.41. *The following are properties of the Lebesgue integral of nonnegative measurable functions.*

1. *Let $f : \mathbb{R} \to [0, \infty]$ be measurable. Then*

$$\int_A f \ge 0, \forall A \subset \mathbb{R} \textit{ measurable.}$$

2. *Let $g : \mathbb{R} \to [0, \infty]$, g measurable, $f \le g$ almost everywhere. Then*

$$\int f \le \int g.$$

3. *If $A \subset B$, then*

$$\int_A f \le \int_B f.$$

1.7.2 Monotone Convergence Theorem

Here's a famous result of measure theory.

Theorem 1.42 (Monotone convergence theorem)**.** *Let $f_n : A \to [0, \infty]$ be a sequence of measurable functions such that*

$$f_1 \le f_2 \le \dots .$$

Let $f = \lim_{n\to\infty} f_n$. Then

$$\int_A f = \lim_{n\to\infty} \int_A f_n.$$

Proof. Since $f_n \leq f_{n+1}$, $\int_A f_n \leq \int_A f_{n+1}$, so $\lim_{n\to\infty} \int_A f_n$ exists, possibly infinite. Since f dominates each f_n, we have

$$\int_A f_n \leq \int_A f,$$

so we conclude $\lim_{n\to\infty} \int_A f_n \leq \int_A f$. Conversely,

$$\int_A f = \sup\left\{\int_A \varphi \mid \varphi \text{ is simple, } 0 \leq \varphi \leq f\right\}$$

. Fix a simple φ, and let $\delta \in [0, 1)$. For $n \in \mathbb{Z}^+$, let $A_n = \{x \in \mathbb{R} \mid f_n(x) > \delta\varphi(x)\}$. Then $A = \bigcup_{n=1}^{\infty} A_n$, since $f(x) = \lim_{n\to\infty} f_n(x), \forall x \in A$. Further, $A_n \subset A_{n+1} \subset \dots$. Now,

$$\begin{aligned}
\int_A f_n &\geq \int_A f_n \chi_{A_n} \\
&\geq \delta \int_A \varphi \chi_{A_n} \\
&= \delta \int_{A \cap A_n} \sum_{i=1}^{k} c_i \chi E_i \\
&= \delta \sum_{i=1}^{k} c_i \mu(A \cap E_i \cap E_n) \\
\implies \lim_{n\to\infty} \int_A f_n &\geq \delta \sum_{i=1}^{k} c_i \mu(A \cap E_i) \\
&= \delta \int_A \varphi.
\end{aligned}$$

Taking the supremum over φ tells us that

$$\lim_{n\to\infty} \int_A f_n \geq \delta \int_A f.$$

Taking $\delta \to 1$,

$$\lim_{n\to\infty} \int_A f_n \geq \int_A f.$$

□

2 Chapter 2. Integration

2.1 Lecture 8. Fatou's Lemma, Lebesgue Integrals of Measurable Functions

Corollary 2.0.1 (Theorem 1.42)**.** *If f_n is simple, with $f_1 \leq f_2 \leq \dots$, and $f_n \to f$ pointwise, then*

$$\int_A f = \lim_{n\to\infty} \int_A f_n.$$

Corollary 2.0.2 (Theorem 1.42)**.** *If $f, g : \mathbb{R} \to [0, \infty]$ is measurable, $c \in \mathbb{R}^+$, then*

1. $\int_A (f + g) = \int_A f + \int_A g, \forall A$ *meausurable,*

2. $\int_A cf = c\int_A f$.

Proof. Let $(f_n)_n, (g_n)_n$ be sequences of simple functions, such that $f_1 \leq f_2 \leq \ldots$, and $g_1 \leq g_2 \leq \ldots$, with $f_n \to f$, $g_n \to g$ pointwise. Then $(f_n + g_n)$ is a sequence of measurable simple functions, such that $f_n + g_n \to f + g$ pointwise. Therefore,

$$\begin{aligned}\int_A f + g &= \lim_{n\to\infty} \int_A f_n + g_n \\ &= \lim_{n\to\infty} \int_A f_n + \int_A g_n \\ &= \lim_{n\to\infty} \int_A f_n + \lim_{n\to\infty} \int_A g_n \\ &= \int_A f + \int_A g.\end{aligned}$$

□

Corollary 2.0.3 (Theorem 1.42)**.** *Suppose $f_n : \mathbb{R} \to [0,\infty]$ is measurable, for all $n \in \mathbb{N}$. Then*

$$\int_A \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_A f_k, \quad \forall A \text{ measurable.}$$

Proof. Set $g_n = \sum_{k=1}^{n} f_n$. Then observe that $g_n \to \sum_{k=1}^{\infty} f_k$, and g_n is a monotone sequence. By the MCT,

$$\begin{aligned}\lim_{n\to\infty} \int_A g_n &= \int_A \lim_{n\to\infty} g_n \\ \implies \sum_{n=1}^{\infty} \int_A g_n &= \int_A \sum_{n=1}^{\infty} f_n\end{aligned}$$

□

2.1.1 Fatou's Lemma

Theorem 2.1 (Fatou's lemma)**.** *Let $f_n : \mathbb{R} \to [0,\infty]$ be measurable, for all $n \in \mathbb{N}$. Then*

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n,$$

for all A measurable.

Proof. Let $g_n = \inf_{k\geq n} f_k$. Then g_n is measurable and monotonically increasing. Note that $\lim_{n\to\infty} g_n = \liminf_{m\to\infty} f_m$. Then by the MCT,

$$\int_A \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int_A g_n$$

We note that $g_n \leq f_n$, so

$$\int_A g_n \leq \int_A f_n.$$

Taking lim infs,

$$\liminf_{n\to\infty} \int_A g_n \leq \liminf_{n\to\infty} \int_A f_n.$$

□

If we have a lemma, we should aim to prove a theorem. This theorem happens to be Lebesgue's dominated convergence theorem.

2.1.2 The Lebesgue Integral

Definition 2.2. Let $f : \mathbb{R} \to \mathbb{R}$. Define $f^+ = \frac{|f|+f}{2}$, $f^- = \frac{|f|-f}{2}$. Note that $f^+, f^- \geq 0$, and $f = f^+ - f^-$. Then define if $\int f^+ < \infty$, and $\int f^- < \infty$, we say f is integrable, and we set

$$\int f = \int f^+ - \int f^-.$$

If $A \subset \mathbb{R}$ is measurable, then we set

$$\int_A f = \int f\chi_A.$$

Theorem 2.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Then*

1. *$f + g$ is Lebesgue integrable, and*

$$\int_A (f+g) = \int_A f + \int_A g, \quad \forall A \text{ measurable.}$$

2. *If $c \in \mathbb{R}$, then cf is Lebesgue integrable, and*

$$\int_A cf = c\int_A f, \quad \forall A \text{ measurable.}$$

3. *The same triangle inequality for Riemann integrals holds:*

$$\left|\int_A f\right| \leq \int_A |f|.$$

Corollary 2.3.1. *If $f : \mathbb{R} \to \mathbb{R}$ is integrable, then*

$$\int_{A\cup B} = \int_A f + \int_B f,$$

if A, B are measurable disjoint sets.

Proof. If $\varphi = f\chi_A, \psi = f\chi_B$, then since $A \cap B = \emptyset$, we have

$$\begin{aligned} \varphi + \psi &= f\chi_{A\cup B} \\ \implies \int \varphi + \int \psi &= \int f\chi_{A\cup B} \\ \implies \int_{A\cup B} f &= \int_A f + \int_B f. \end{aligned}$$

□

Definition 2.4. A proposition $P(x)$ depending on $x \in \mathbb{R}$ holds almost everywhere if the set M for which $P(x)$ does not holds for all $x \in M$ has measure 0.

2.2 Lecture 9. Lebesgue's Dominated Convergence Theorem

Theorem 2.5 (Dominated Convergence Theorem)**.** *Let $A \subset \mathbb{R}$ be a measurable set, and $f_n : A \to \mathbb{R}$ be measurable functions for all $n \in \mathbb{N}$. Suppose $f : A \to \mathbb{R}$ is such that $f_n \to f$ pointwise almost everywhere. Suppose $g : A \to [0, \infty)$ is integrable, with $|f_n(x)| \leq g(x)$ for almost all $x \in A$, for all $n \in \mathbb{N}$. Then*

$$\int_A f_n \to \int_A f.$$

Proof. Set $E_1^{(n)} = \{x \in A \mid f_n(x) \not\to f(x)\}$. Set $E_2^{(n)} = \{x \in A \mid |f_n(x)| > f(x)\}$. By assumption, these sets each have measure 0. Setting $E_2 = \bigcup_{n=1}^{\infty} E_2^{(n)}$, we know

$$\lambda(E_2) \leq \sum_{n=1}^{\infty} \lambda(E_2^{(n)}) = 0,$$

so $\lambda(E_2) = 0$. We can similarly set $E_1 = \bigcup_{n=1}^{\infty} E_1^{(n)}$, and deduce $\lambda(E_1) = 0$. Then set $E = E_1 \cup E_2$, so and we will find $\lambda(E) = 0$. It suffices to show that

$$\int_{A \setminus E} f_n \to \int_{A \setminus E} f.$$

We have $f_n(x) \to f$, $\forall x \in A \setminus E$, and $|f_n(x)| \leq g(x), \forall x \in A \setminus E$. We can rewrite the bounding of f_n as

$$\begin{aligned} -g(x) \leq f_n(x) \leq g(x), \quad \forall x \in A \setminus E \\ \implies f_n(x) + g(x) \geq 0, \quad \forall x \in A \setminus E. \end{aligned}$$

Apply Fatou's lemma:

$$\begin{aligned} \int_{A \setminus E} \liminf_{n \to \infty} (f_n + g) &\leq \liminf_{n \to \infty} \int_{A \setminus E} (f_n + g) \\ \implies \int_{A \setminus E} f + \int_{A \setminus E} g &\leq \liminf_{n \to \infty} \int_{A \setminus E} f_n + \int_{A \setminus E} g \\ \implies \int_{A \setminus E} f &\leq \liminf_{n \to \infty} \int_{A \setminus E} f_n. \end{aligned}$$

On the other hand,

$$g(x) - f_n(x) \geq 0, \quad \forall x \in A \setminus E.$$

Apply Fatou's lemma once more:

$$\begin{aligned} \liminf_{A \setminus E} \int_{A \setminus E} (g - f_n) &\leq \liminf_{n \to \infty} \int_{A \setminus E} (g - f_n) \\ \implies \int_{A \setminus E} g - \int_{A \setminus E} f &\leq \int_{A \setminus E} g + \liminf_{n \to \infty} \int_{A \setminus E} (-f_n) \\ \implies \int_{A \setminus E} f &\geq \limsup_{n \to \infty} \int_{A \setminus E} f_n. \end{aligned}$$

Altogether,

$$\int_{A \setminus E} f \leq \liminf_{n \to \infty} \int_{A \setminus E} f_n \leq \limsup_{n \to \infty} \int_{A \setminus E} f_n \leq \int_{A \setminus E} f.$$

Therefore,

$$\liminf_{n\to\infty} \int_{A\setminus E} f_n = \limsup_{n\to\infty} \int_{A\setminus E} f_n,$$

so $\lim_{n\to\infty} \int_{A\setminus E} f_n$ exists, and

$$\lim_{n\to\infty} \int_{A\setminus E} f_n = \int_{A\setminus E} f.$$

□

2.2.1 A Particular Banach Space

Definition 2.6. Let $A \subset \mathbb{R}$ be a measurable set. Fix $p \in [1, \infty]$. For now, assume $p < \infty$. We will deal with $p = \infty$ soon. Define

$$L^p(A) = \{f : A \to \mathbb{R} \mid |f|^p \text{ is integrable}\}. \tag{*}$$

Lemma 2.7. *The space $L^p(A)$ is a vector space under the ordinary function addition and scalar multiplication by scalars in $\mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$, $f \in L^p(A)$. If $\int_A |f|^p < \infty$, then

$$\int_A |cf|^p = \int_A |c|^p |f^p| = |c|^p \int_A |f^p| < \infty.$$

Let $g \in L^p(A)$. Then

$$\begin{aligned} \int_A |f+g|^p &\leq \int_A ||f| + |g||^p \\ &\overset{(1)}{\leq} 2^{p-1} \int_A |f|^p + |g|^p < \infty, \end{aligned}$$

where (1) falls as a result of the convexity of $x \mapsto x^p$, on $[0, \infty)$, and is left as an exercise. □

We say the elements of $L^p(A)$ are p-integrable functions. Usually we say in the $p = 1$ case, that the functions are integrable, and in the $p = 2$ case we say they are square-integrable.

Recall that a norm on a vector space V is a function, written $\|\cdot\|$ traditionally, $\|\cdot\| : V \to [0, \infty)$, that is positive definite, homogeneous, and satisfies the triangle inequality.

Theorem 2.8. *Let $\|\cdot\| : L^p(A) \to [0, \infty)$, such that*

$$\|f\|_p = \left(\int_A |f|^p\right)^{1/p}.$$

Then $\|\cdot\|$ is a norm on $L^p(A)$.

At this point this definition should immediately alarm us. Every function which is 0 almost everywhere has integral 0, so this can't be a norm. Worry not. When we talk about the space $L^p(A)$, we treat it as a quotient space by the functions which are zero almost everywhere. Specifically, consider $L^P(A)$ to be the space of equivalence classes of functions within the definition (*), under the equivalence relation $f, g \in L^p(A), f \equiv g$ if $f = g$ almost everywhere.

But notably when working with functions in L^p we almost never refer to them as equivalence classes. We will ordinarily treat them as functions as expected, but for now we need to treat these functions as equivalence classes.

Lemma 2.9. *If $f : \mathbb{R} \to [0, \infty)$ is such that $\int f = 0$, then $f = 0$ almost everywhere.*

Proof. Let $B_n = \{x \in \mathbb{R} \mid f(x) \geq \frac{1}{n}\}$. Then

$$0 = \int f \geq \int_{B_n} f \geq \int_{B_n} \frac{1}{n} = \frac{1}{n}\mu(B_n).$$

Conclude $\mu(B_n) = 0$. Thus $\mu(\bigcup_{n\in\mathbb{N}} B_n) = 0$, but $\bigcup_{n\in\mathbb{N}} B_n = \{x \mid f(x) \neq 0\}$. □

Corollary 2.9.1. *If $f \in L^p(A)$, then $\|f\|_p = 0$ iff $f = 0$ almost everywhere.*

Next time, we will show that $\|\cdot\|_p$ is a norm by showing the remainder of the properties hold.

3 Chapter 3. Lp Spaces

3.1 Lecture 10. Young's Inequality, Hölder's Inequality, Minkowski's Inequality

Theorem 3.1 (Young's inequality)**.** *Suppose $a, b \in \mathbb{R}$, $a, b \geq 0$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $p, q \in (1, \infty)$,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 3.2 (Hölder's inequality)**.** *Suppose $f, g : A \to \mathbb{R}$ for $A \subset \mathbb{R}$ Suppose that $\frac{1}{p} + \frac{1}{q} = 1$, for $1 < p, q < \infty$. Then*

$$\int_A |fg| \leq \|f\|_p \|g\|_q .$$

Proof. It suffices to prove that

$$\int_A \frac{|f|}{\|p\|} \frac{|g|}{\|g\|_q} \leq 1.$$

By Young's inequality,

$$\begin{aligned}
\int_A \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} &\leq \int_A \frac{1}{p} \frac{|f|^p}{\|f\|^p} + \frac{1}{q} \frac{|g|^q}{\|g\|^q} \\
&= \frac{1}{p} \frac{1}{\|f\|^p} \int_A \|f\|^p + \frac{1}{q} \frac{1}{\|g\|^q} \int_A \|g\|^q \\
&= 1.
\end{aligned}$$

□

Theorem 3.3. *If $f, g \in L^p(A)$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. We have

$$\begin{aligned}
\|f+g\|_p^p &= \int_A |f+g|^p = \int_A |f+g|^{p-1}|f+g| \\
&\leq \int_A |f+g|^{p-1}|f| + \int_A |f+g|^{p-1}|g| \\
&\leq \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \left\|(f+g)^{p-1}\right\|_q .
\end{aligned}$$

Consider that

$$\begin{aligned}
\left\|(f+g)^{p-1}\right\|_q &= \left(\int_A \left|(f+g)^{(p-1)q}\right|\right)^{1/q} \\
&= \left(\int_A |(f+g)^p|\right)^{1/q} \\
&= \|f+g\|_p^{p/q} \\
\implies \|f+g\|_p^p &\leq \|f+g\|_p^{p/q}\left(\|f\|_p + \|g\|_p\right) \\
\implies \|f+g\|_p^{p-p/q} &\leq \|f\|_p + \|g\|_p
\end{aligned}$$

But $1+\frac{p}{q}=p$, so $p-\frac{p}{q}=1$, and thus

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

□

Definition 3.4. A Banach space is a complete normed space.

Definition 3.5. A series $\sum_{n=1}^{\infty} x_n$ in a normed space X converges iff its sequence of partial sums $\{S_n\}_n$ converges. The series is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges.

Theorem 3.6. *Let X be a normed space. Then it is a Banach space iff every absolutely convergent series is convergent.*

The proof will be homework.

4 Chapter 4. General Measures

4.1 Lecture 12. Integrals Over General Measure Spaces

Let $A=(a,b)\subset\mathbb{R}$. Recall that we defined

$$\|f\|_\infty = \inf_{\lambda(E)=0}\ \sup_{x\in A\setminus E} |f(x)| = \text{ess sup}_A f.$$

Lemma 4.1. *For $f\in L^\infty(A)$, $N=\{x\in A \mid |f(x)|>\|f\|_\infty\}$ has measure zero.*

Theorem 4.2. *The function $\|\cdot\|_\infty$ is a norm on $L^\infty(A)$.*

Proof. Let $f, g \in L^\infty(A)$. Let $M = \{x \in A \mid |f(x)| > \|f\|_\infty\}$, $N = \{x \in A \mid |g(x)| > \|g\|_\infty\}$. Then by our lemma, $\lambda(M) + \lambda(N) = 0$, and $\lambda(M \cup N) = 0$. Consider

$$\begin{aligned} \sup_{x \in A \setminus (M \cup N)} |f(x) + g(x)| &\leq \sup_{x \in A \setminus (M \cup N)} (|f(x)| + |g(x)|) \\ &\leq \sup_{x \in A \setminus (M \cup N)} |f(x)| + \sup_{y \in A \setminus (M \cup N)} |g(y)| \\ &\leq \|f\|_\infty + \|g\|_\infty . \end{aligned}$$

For homogeneity, let $c \in \mathbb{R}$. Then

$$\begin{aligned} \sup_{x \in A \setminus M} |cf(x)| &= \sup_{x \in A \setminus M} |c|\, |f(x)| \\ &\leq |c| \sup_{x \in A \setminus M} |f(x)| \\ \implies |cf|_\infty &\leq |c|\, \|f\|_\infty . \end{aligned}$$

If $c = 0$, we are done, otherwise assume $c \neq 0$, then

$$\|f\|_\infty = \left\| \frac{1}{c}(cf) \right\|_\infty \leq \frac{1}{|c|} \|cf\|_\infty \implies |c|\, \|f\|_\infty \leq \|cf\|_\infty .$$

Our lemma tells us that $\|f\|_\infty = 0$ iff $f = 0$ a.e. □

Proposition 4.3. *For $A = [a, b]$, $L^\infty([a, b]) \subset L^p([a, b])$, for $p > 1$.*

Proof. Let $f \in L^\infty([a, b])$. Then

$$\int_A |f|^p \leq \|f\|_\infty^p \int_A 1 = (b - a) \|f\|_\infty^p .$$

□

Before moving on the general measure spaces, we will quickly define the Lebesgue integral for complex valued functions. Let $f : \mathbb{R} \to \mathbb{C}$. Write $f = g + ih$, where $g = \mathrm{Re}(f), h = \mathrm{Im}(f)$. If g, h are both integrable, then we call f integrable, and let

$$\int f = \int g + i \int h.$$

We note that the triangle inequality for integrals still holds in the complex case:

$$\left| \int f \right| \leq \int |f| .$$

4.1.1 General Measure Spaces

Let X be a set, let $\mathscr{A}$ be a σ-algebra, or σ-field over X. Let $\mu : \mathscr{A} \to [0, \infty]$ be a function satisfying

1. $\mu(\emptyset) = 0$,
2. for $A_i \in \mathscr{A}$, $i \in \mathbb{N}$, $A_i \cap A_j = \emptyset$, $i \neq j$, we have

$$\mu\left(\bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty \mu(A_i).$$

Then we call μ a measure, and we call $(X, \mathscr{A})$ a measurable space, and $(X, \mathscr{A}, \mu)$ a measure space.

Given $f : X \to [0, \infty]$, we define

$$\int f \, d\mu = \sup \left\{ \int g \mid 0 \leq g \leq f, g \text{ simple} \right\},$$

where a simple function is just as we have defined in the Lebesgue sense:

$$g = \sum_{i=1}^{n} c_i \chi_{A_i}, \quad A_1, \ldots, A_n \in \mathscr{A},$$

and

$$\int g \, d\mu = \sum_{i=1}^{n} c_i \mu(A_i).$$

And for $f : X \to [-\infty, \infty]$, we take $f = f^+ f^-$, and write

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

It turns out that the monotone convergence theorem, Fatou's lemma, and Lebesgue's dominated convergence theorem all hold with general measures. So what can we do with general measures?

Definition 4.4. A measure space $(X, \mathscr{A}, \mu)$ is called σ-finite if there exists $(A_n)_{n=1}^{\infty}$ such that

$$A_1 \subset A_2 \subset \ldots,$$

satisfying

$$\bigcup_{n=1}^{\infty} A_n = X,$$

and

$$\mu(A_n) < \infty, \forall n \in \mathbb{N}.$$

We can clearly see that the Lebesgue measure induces a σ-finite measurable space on $\mathbb{R}$. Consider that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n].$$

4.1.2 Product Measures

In general, $\mathbb{R}^n$ has a Lebesgue measure. It is much more difficult to construct this Lebesgue measure if we were to replicate our construction in $\mathbb{R}$. However, there is an easy alternative we can consider when we study product measures.

Theorem 4.5. *Let $(X, \mathscr{A}, \mu), (Y, \mathscr{B}, \nu)$ be measure spaces. Then $(X \times Y, \mathscr{A} \otimes \mathscr{B}, \tau)$ is a measure space, where $\mathscr{A} \otimes \mathscr{B}$ is the σ-field generated by $A \times B$, for $A \in \mathscr{A}, B \in \mathscr{B}$, and $\tau(A, B) = \mu(A)\nu(B)$, for $(A, B) \in \mathscr{A} \otimes \mathscr{B}$.*

Proof. Consider

$$\mathcal{R} = \left\{ \bigcup_{i=1}^{k} A_i \times B_i, A_i \in \mathscr{A}, B_i \in \mathscr{B}, k \in \mathbb{N} \right\}.$$

Then $\mathcal{R}$ is closed under finite unions and complements. It is closed under complement, as

$$(A \times B)^C = (A^C \times Y) \bigcup (A \times B^C).$$

On $\mathcal{R}$, take $R \in \mathcal{R}$, and write

$$R = \bigcup_{i=1}^{k} A_i \times B_i, \text{ as a disjoint union.}$$

Define $\tau_R = \sum_{i=1}^{k} \mu(A_i)\nu(B_i)$. Then we claim that there exists an extension of τ_R to $\mathscr{A} \otimes \mathscr{B}$, by the theorem below. We call this extension τ our product measure. □

Theorem 4.6. *If $\mathcal{R}$ is an algebra of sets, and $\mathscr{A}$ is a σ-algebra generated by $\mathcal{R}$. If $\pi : \mathcal{R} \to [0, \infty]$ is a measure, then there exists $\tilde{\pi} : \mathscr{A} \to [0, \infty]$, such that $\tilde{\pi}_R = \pi$. If π is σ-finite, then $\tilde{\pi}$ is unique.*

Example 4.7. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. Then $\lambda \times \lambda : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$ is a measure. This measure generalizes the notion of area on the plane.

Conjecture 1. *Kakeya's conjecture: hat is the region of smallest possible area in which it is possible to rotate a needle 180 degrees in the plane? Such regions are called Kakeya needle sets.*

This conjecture was solved recently in 2025, and of course, involves the study of two dimensional Lebesgue measure.

4.1.3 Fubini's Theorem

Theorem 4.8 (Fubini's Theorem)**.** *For nonnegative $f : X \times Y \to [0, \infty]$,*

$$\iint_{X \times Y} f(x, y)\, d(\mu \times \nu)(x, y) = \int_X \int_Y f(x, y)\, d\mu\, d\nu = \int_Y \int_X f(x, y)\, d\nu\, d\mu.$$

5 Chapter 5. Complex Analysis

5.1 Introduction

I skip ahead a bit. I'm following mostly Alfhor's *Complex Analysis.*

Exercise 5.1 (Lagrange's identity)**.** Prove Lagrange's identity:

$$\left| \sum_{i=1}^{n} a_i b_i \right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \le i < j \le n} \left| a_i \overline{b_j} - a_j \overline{b_i} \right|^2.$$

Proof. Proof is by induction, and it turns out to be easier to work right-to-left. Consider

$$\begin{aligned}
&|a_1 b_1|^2 + |a_1 b_2|^2 + |a_2 b_1|^2 + |a_2 b_2|^2 - \left| a_1 \overline{b_2} - a_2 \overline{b_1} \right|^2 \\
&= |a_1 b_1|^2 + \cancel{|a_1 b_2|^2} + \cancel{|a_2 b_1|^2} + |a_2 b_2|^2 - \cancel{|a_1 b_2|^2} - \cancel{|a_2 b_1|^2} + 2\,\mathrm{Re}(a_1 \overline{a_2} b_1 \overline{b_2}) \\
&= |a_1 b_1 + a_2 b_2|^2.
\end{aligned}$$

Assume that

$$\left|\sum_{i=1}^{n-1} a_i b_i\right|^2 = \sum_{i=1}^{n-1} |a_i|^2 \sum_{i=1}^{n-1} |b_i|^2 - \sum_{1\le i<j\le n-1} \left|a_i\overline{b_j} - a_j\overline{b_i}\right|^2 .$$

We play the same game by massaging the rhs of our desired inequality:

$$\begin{aligned}
&\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1\le i<j\le n} \left|a_i\overline{b_j} - a_j\overline{b_i}\right|^2 \\
&= \sum_{i=1}^{n-1} |a_i|^2 \sum_{i=1}^{n-1} |b_i|^2 - \sum_{1\le i<j\le n-1} \left|a_i\overline{b_j} - a_j\overline{b_i}\right|^2 \\
&+ |a_n|^2 \sum_{i=1}^{n-1} |b_i|^2 + |b_n|^2 \sum_{i=1}^{n-1} |a_i|^2 + |a_n b_n|^2 \\
&- \sum_{i=1}^{n-1} \left|a_n\overline{b_i} - a_i\overline{b_n}\right|^2 \\
&= \left|\sum_{i=1}^{n-1} a_i b_i\right|^2 \\
&+ \cancel{|a_n|^2 \sum_{i=1}^{n-1} |b_i|^2} + \cancel{|b_n|^2 \sum_{i=1}^{n-1} |a_i|^2} + |a_n b_n|^2 \\
&- \sum_{i=1}^{n-1} \left(\cancel{|a_n b_i|^2} + \cancel{|a_i b_n|^2} - 2\,\mathrm{Re}(\overline{a_i} a_n \overline{b_i} b_n)\right) \\
&= \left|\sum_{i=1}^{n-1} a_i b_i\right|^2 + |a_n b_n|^2 + \sum_{i=1}^{n-1} 2\,\mathrm{Re}(\overline{a_i} a_n \overline{b_i} b_n) \\
&= \left|\sum_{i=1}^{n-1} a_i b_i\right|^2 + |a_n b_n|^2 + 2\,\mathrm{Re}\left(\sum_{i=1}^{n-1} \overline{a_i} a_n \overline{b_i} b_n\right) \\
&= \left|\sum_{i=1}^{n-1} a_i b_i\right|^2 + |a_n b_n|^2 + 2\,\mathrm{Re}\left(a_n b_n \sum_{i=1}^{n-1} \overline{a_i b_i}\right) \\
&= \left|\sum_{i=1}^{n} a_i b_i\right|^2 .
\end{aligned}$$

□

5.2 Inequalities

There is no order relation in the complex-number system. Inequalities in complex analysis hence must be between real numbers. From the definition of absolute value, we deduce

$$\begin{aligned}
-|a| &\le \mathrm{Re}\, a \le |a| \\
-|a| &\le \mathrm{Im}\, a \le |a| .
\end{aligned}$$

For example, to bound $\operatorname{Re} a$, consider that if $a = x + yi$,

$$\begin{aligned}(\operatorname{Re} a)^2 &= x^2 \le x^2 + y^2 = |a|^2 \\ \implies |\operatorname{Re} a| &\le |a| .\end{aligned}$$

Since we know for real numbers,

$$-|\operatorname{Re} a| \le \operatorname{Re} a \le |\operatorname{Re} a| ,$$

our bound is given by negating $|\operatorname{Re} a| \le |a|$. The proof is analogous for $\operatorname{Im} a$.

We achieve equality $\operatorname{Re} a = |a|$ iff a is real, and $x \ge 0$. This much is clear: $x = \sqrt{x^2} = |a|$.

Applying our bounds for $\operatorname{Re} a$ tells us

$$|a+b|^2 = |a|^2 + |b|^2 + 2 \operatorname{Re} ab \le |a|^2 + |b|^2 + 2|a|\,|b| = (|a| + |b|)^2.$$

Our pops our *triangle inequality*:

$$|a+b| \le |a| + |b| ,$$

which for the modulus function, we will motivate geometrically later. Of course, by induction,

$$|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n| .$$

5.3 Cauchy Residue Theorem

Last time we studied Laurent series, the complex analogues to Taylor series:

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

Let $R_1 := \limsup_{n \to -\infty} |c_n|^{1/n}$. Let $R_2 = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$. By our previous discussion, if $R_1 < R_2$, then the Laurent series is convergent in the region $R_1 < |z-a| < R_2$. Also, if $\delta > 0$, the Laurent series is absolutely convergent in $R_1 + \delta < |z-a| < R_2 - \delta$. (Of course, δ is such that $R_1 + \delta < R_2 - \delta$.)

In the same ring, we have uniform convergence for the function $f_n(z) = \sum_{k=-n}^{n} c_k (z-a)^k$, that is, $f_n \to f := \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ uniformly.

Differentiation. On this small δ-ring, we can differentiate term-by-term, since we have uniform convergence:

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n (z-a)^{n-1}.$$

There are many nice results we've covered about analytic functions. Here is another.

Theorem 5.1 (Laurent expansion)**.** *Let $0 \le R_1 < R_2$. Let f be an analytic function on the set $\Omega := \{R_1 < |z-a| < R_2\}$. Then f has a Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad \textit{in } \Omega.$$

We will use two theorems we have proven previously to prove this. Recall Cauchy's theorem: $f : \Omega \to \mathbb{C}$, γ a closed curve in Ω implies $\int_\gamma f(z)\, dz = 0$. Recall Cauchy's integral formula: for the same f, γ, Ω, we have $\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) n(\gamma, z)$, where $n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma d\zeta / (\zeta - z)$.

We also would like to make formal a notion which we have become familiar with. Recall that sums of curves $\gamma_1, \ldots, \gamma_n$ are the unions of curves: $\gamma_1 \cup \ldots \cup \gamma_n$.

Definition 5.2. A *cycle* is a sum of finitely many closed curves.

Definition 5.3. Let γ be a *cycle*, $\gamma = \gamma_1 + \ldots + \gamma_n$, where each γ_i is a closed curve. The *index* of a point p with respect to a cycle, $n(\gamma, p)$ is defined to be

$$n(\gamma, p) = \sum_{i=1}^{n} n(\gamma_i, p).$$

Naturally, let us extend Cauchy's theorem and Cauchy's integral formula to cycles.

Proposition 5.4. *Let $f : \Omega \to \mathbb{C}$ be analytic, and let $\gamma = \gamma_1 + \ldots + \gamma_n$, where each γ_i is a closed curve. Then the following hold.*

1. $\int_\gamma f(z)\, dz = 0$.
2. $n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \forall z \in \Omega \setminus \gamma$, *if* $n(\gamma, p) = 0$, *for all* $p \notin \Omega$.

Proof (Theorem 5.1). Let γ be the cycle defined by adding γ_1 the ring defined by R_1, and γ_2 the ring defined by R_2, both postively oriented. Then $z \in \Omega$ has index 1, so we can use our extended Cauchy's integral formula as follows:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The second integral reduces to a Taylor expansion:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z - a)} = \frac{1}{z - a}\left(1 - \frac{z - a}{\zeta - a}\right).$$

Consider an auxilary variable:

$$\omega = \frac{1}{z - a}.$$

Note that $|\omega| < \frac{1}{R_1} \iff |z - a| > R - 1$. The function $g(\omega) = f(a + \frac{1}{\omega})$ is analytic in $0 < |\omega| < \frac{1}{R_1}$. We note that g has a removable singularity at 0:

$$\lim_{n \to 0} |\omega||g(\omega)| = 0,$$

since as $\omega \to 0$, $|g(\omega)| \to 0$.

Thus, we can assume that $g(\omega) = c_0 + c_1\omega + c_2\omega^2 + \ldots$, and thus

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = c_0 + \frac{c_1}{z - a} + \frac{c_2}{(z - a)^2} + \ldots,$$

on $|z - a| > R_1$.

□